

Generalized Weierstrass formulae, soliton equations and Willmore surfaces

I. Tori of revolution and the mKDV equation

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Abstract: A new approach is proposed for study structure and properties of the total squared mean curvature W of surfaces in \mathbf{R}^3 . It is based on the generalized Weierstrass formulae for inducing surfaces. The quantity W (Willmore functional or extrinsic Polyakov action) is shown to be invariant under the modified Novikov–Veselov hierarchy of integrable flows.

It is shown that extremals of W (Willmore surfaces) obey the two-dimensional Schrodinger equation and possible relations between Willmore surfaces and the Novikov–Veselov hierarchy of integrable equations are discussed.

The 1+1-dimensional case and, in particular, Willmore tori of revolution, are studied in details. The Willmore conjecture is proved for the mKDV-invariant Willmore tori.

1. Introduction

Surfaces and their dynamics play an important role in many interesting phenomena both in classical and quantum physics (see, e.g., [1, 2, 3, 4]). In the string theory and 2D-gravity based on the Polyakov integral over surfaces ([5]) contributions from certain special classes of surfaces are of crucial importance.

In mathematics the foundations of differential geometry of surfaces have been basically completed almost one century ago (see, e.g., [6, 7]). Nevertheless, number of problems concerning the structure and properties of special classes of surfaces are still remain open (see, e.g., [8, 9, 10]).

The total squared mean curvature

$$W = \int H^2 d\mu \quad (1.1)$$

is one of the most important characteristics of surfaces in the three-dimensional Euclidean space \mathbf{R}^3 . Here H is a mean curvature and $d\mu$ is a Liouville measure with respect to metric induced by immersion. In string theory and $2D$ -gravity the functional W is known as the Polyakov's extrinsic action. The properties of the Polyakov's action were the subject of study in a number of papers (see the review [11]).

In the differential geometry the functional W has been investigated within the study of so-called Willmore surfaces ([9, 10, 12, 13, 14, 15, 16, 17, 18, 19]). These surfaces provide extremum to W and obey the corresponding Euler–Lagrange equation

$$\Delta H + 2H(H^2 - K) = 0 \quad (1.2)$$

where K is the Gaussian curvature and Δ is the Laplace–Beltrami operator. A study of extremals of the functional W is also of great importance for $2D$ -gravity. In quantum theory they correspond to so-called zero modes.

In the present paper we propose a new approach to study structure and properties of the functional W (1.1) and corresponding extremals of (1.2). Our approach is based on the generalization of the Weierstrass formulae proposed in [20]. This generalization allows one to induce generic surfaces in \mathbf{R}^3 via solutions of the system of two linear equations. We show that the generalized Weierstrass inducing from [20] is equivalent to earlier extension of Weierstrass formulae given in [21] (see also [22, 23]). An advantage of new formulation is that it allows to describe and construct integrable deformations of induced surfaces via the modified Novikov–Veselov (mNV) hierarchy flows ([24], for Veselov–Novikov equation see [25], and also [26]). It is shown that the functional W is invariant under mNV deformations.

In this paper (part I) we restrict ourselves to the $1 + 1$ -dimensional limit of our scheme. We concentrate on the mKDV-invariant Willmore tori of revolution. Properties of such tori are studied. For mKDV-invariant Willmore tori of revolution we prove the Willmore conjecture that their total squared mean curvature $W \geq 2\pi^2$ and, moreover, $W = 2\pi^2$ only for the Clifford torus.

The relation of this problem to soliton theory was noticed about ten years ago and the finite-gap integration was used for constructing Willmore tori (see [16, 18, 19]). But the formulae for finite-gap solutions is rather unefficient. We will discuss only Willmore tori of revolution postponing the consideration of general case and will not give rigorous mathematical proofs (but notice that that is possible as one will see). Our main goal is to shed light on the one-dimensional case for using this approach to the main problem.

The paper is organized as follows. In section 2 the generalized Weierstrass inducing is discussed. The equivalence of equation (1.2) to the 2D-Schrodinger equation is proved in section 3. The $1 + 1$ -dimensional case is considered in section 4. The construction of the Clifford torus via the generalized Weierstrass inducing is given in section 5. In section 6 we prove the Willmore conjecture for the mKDV-invariant Willmore tori of revolution.

2. Generalized Weierstrass inducing

The generalization of the Weierstrass inducing, proposed in [20], starts with the linear system

$$\begin{aligned}\psi_{1z} &= p\psi_2, \\ \psi_{2\bar{z}} &= -p\psi_1\end{aligned}\tag{2.1}$$

where $p(z, \bar{z})$ is a real-valued function, ψ_1 and ψ_2 are complex-valued functions of the complex variable z , and bar denotes the complex conjugation. Then one can define three functions $(X^1(z, \bar{z}), X^2(z, \bar{z}), X^3(z, \bar{z}))$ as follows

$$\begin{aligned}X^1 + iX^2 &= 2i \int_{\Gamma} (\bar{\psi}_1^2 dz' - \bar{\psi}_2^2 d\bar{z}'), \\ X^1 - iX^2 &= 2i \int_{\Gamma} (\psi_2^2 dz' - \psi_1^2 d\bar{z}'), \\ X^3 &= -2 \int_{\Gamma} (\psi_2 \bar{\psi}_1 dz' + \psi_1 \bar{\psi}_2 d\bar{z}')$$

where Γ is an arbitrary curve and treat X^1, X^2 , and X^3 as the coordinates of surface in \mathbf{R}^3 while z, \bar{z} are local coordinates on the surface.

An arbitrary surface in \mathbf{R}^3 with non-vanishing mean curvature can be represented by using this method (that was proved for Kenmotsu representation ([21, 22]) and valid for (2.2) since this representation is equivalent to Kenmotsu's one as it will be shown below).

The metric which is induced by this immersion is equal to ([20])

$$4u(z, \bar{z})^2 dz d\bar{z}\tag{2.3}$$

where $u(z, \bar{z}) = |\psi_1|^2 + |\psi_2|^2$, and the Gaussian curvature K and the mean curvature H have the form

$$K = -\frac{1}{4u^2} \Delta \log u, \quad H = \frac{p}{u}.\tag{2.4}$$

It is easy to notice that formulae (2.1) and (2.2) are reduced to those of Weierstrass in the case $p = 0$, i.e., for minimal surfaces.

Exact explicit solutions to (2.1) provide us surfaces in \mathbf{R}^3 via (2.2) by quadratures. A family of surfaces parametrized by an arbitrary number of holomorphic functions has been constructed in [20].

Another extension of the Weierstrass inducing to nonminimal surfaces has been proposed earlier by Kenmotsu ([21], see also [22]). In this approach a surface in \mathbf{R}^3 is defined :

$$X^i = \int \omega_i\tag{2.5}$$

where

$$\omega_1 = \text{Re}\{\phi(1 - f^2)dz\},$$

$$\omega_2 = \text{Re}\{i\phi(1 + f^2)dz\}, \quad (2.6)$$

$$\omega_3 = \text{Re}\{2\phi f dz\}$$

and functions f and ϕ obey the relation

$$(\log \phi)_{\bar{z}} = -\frac{2\bar{f}f_{\bar{z}}}{1 + |f|^2}. \quad (2.7)$$

The mean curvature has the form

$$H = -\frac{\bar{f}_{\bar{z}}}{\phi(1 + |f|^2)}.$$

The generalized Weierstrass inducing (2.1) — (2.2) and the Kenmotsu inducing (2.5) — (2.7) are equivalent to each other. The relation between ψ_1, ψ_2, p and f, ϕ are of the form (see also [23])

$$f = i\frac{\bar{\psi}_1}{\psi_2}, \phi = i\psi_2^2 \quad (2.8)$$

and

$$p = -\frac{(\bar{f}\bar{\phi})_z}{\sqrt{\phi\bar{\phi}}} = \frac{\phi f_{\bar{z}}}{\sqrt{\phi\bar{\phi}}(1 + |f|^2)}. \quad (2.9)$$

One of advantages of the representation (2.1) — (2.2) is that, due to the linear character of (2.1), it allows to describe integrable deformations of induced surfaces in a simple manner. For that purpose one uses the basic idea of the inverse spectral transform method and looks for the deformation of all quantities in (2.1) in time t via the equation

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = A \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (2.10)$$

where A is a differential 2×2 -matrix operator ([20]). The requirement of compatibility of (2.1) and (2.10) first quarantees the preservation in time t of the generalized Weierstrass inducing (2.2) and, second, leads to nonlinear differential equations for p .

The simplest nontrivial example corresponds to nonlinear problem (2.10) of the form

$$\begin{aligned} &(\partial_t + \partial_z^3 + \partial_{\bar{z}}^3 + 3 \begin{pmatrix} 0 & p_z \\ 0 & \omega \end{pmatrix} \partial_z + 3 \begin{pmatrix} \bar{\omega} & 0 \\ p_{\bar{z}} & 0 \end{pmatrix} \partial_{\bar{z}} + \\ &\frac{3}{2} \begin{pmatrix} \bar{\omega}_{\bar{z}} & 2p\omega \\ -2p\bar{\omega} & \omega_z \end{pmatrix})\psi = 0. \end{aligned} \quad (2.11)$$

The associated nonlinear integrable equation for p is

$$p_t + p_{zzz} + p_{\bar{z}\bar{z}\bar{z}} + 3p_z\omega + 3p_{\bar{z}}\bar{\omega} + \frac{3}{2}p\bar{\omega}_{\bar{z}} + \frac{3}{2}p\omega_z = 0, \quad (2.12)$$

$$\omega_{\bar{z}} = (p^2)_z.$$

Equation (2.12) is known as the modified Novikov–Veselov equation (mNV) ([24]). The mNV equation is integrable by the Inverse Scattering Transform (IST) method. The hierarchy of integrable PDEs associated with the system (2.1) arises as the compatibility condition of (2.1) with the system of the form (2.10) where

$$A = -(\partial_z^{2k+1} + \partial_{\bar{z}}^{2k+1} + \sum_{m=1}^{2k-1} (u_m \partial_z^m + \bar{u}_m \partial_{\bar{z}}^m)). \quad (2.13)$$

All equations of mNV hierarchy commute to each other and are integrable by the IST method.

Thus the integrable dynamics of surfaces is induced by the mNV hierarchy via (2.2). This integrable dynamics of surfaces inherits all properties of the mNV hierarchy. Note that the minimal surfaces ($p = 0$) are invariant under such dynamics.

Within the generalized Weierstrass inducing the functional W has a very simple form. Indeed, by using of (2.3) and (2.4), one gets

$$W = 4 \int p^2 dz d\bar{z}. \quad (2.14)$$

Thus, using the exact solutions of system (2.1), one gets W .

An important properties of W is that it is invariant under the mNV deformations. Indeed, rewriting the mNV equation (2.12) in the form

$$(p^2)_t + (2p_{zz} - p_z^2 + 3p^2\omega)_z + (2p_{\bar{z}\bar{z}} - p_{\bar{z}}^2 + 3p^2\bar{\omega})_{\bar{z}} = 0$$

one easily concludes that

$$W_t = 4 \int (p^2)_t dz d\bar{z} = 0 \quad (2.15)$$

for periodic surfaces (p is a periodic function) or for p decreasing as $|z| \rightarrow \infty$. The quantity $\int p^2 dz d\bar{z}$ is the integral of motion for the whole mNV hierarchy (we will give the complete prove for $1 + 1$ -case below).

Thus the infinite-parametric family of mNV deformed surfaces have the same value of W .

3. Willmore surfaces, two-dimensional Schrodinger equation and NV hierarchy

We consider now the Euler–Lagrange equation (1.2) which defines the Willmore surfaces. We will consider a surface in the system of local coordinates

formed by minimal lines. The metric in these coordinates has the form (2.3) where u is some real-valued function. Then let us introduce the real-valued function p via $H = \frac{p}{u}$.

Since in these coordinates

$$H = \frac{H_{z\bar{z}}}{u^2}, K = -\frac{(\log u)_{z\bar{z}}}{u^2}$$

equation (1.2) becomes

$$u_{z\bar{z}} - (\log p)_{\bar{z}} u_z - (\log p)_z u_{\bar{z}} + \frac{p_{z\bar{z}} + 2p^3}{p} u = 0. \quad (3.1)$$

Then we introduce new variable ξ via $u = p\xi$. In terms of ξ equation (3.1) is of the form

$$\xi_{z\bar{z}} + 2((\log p)_{z\bar{z}} + p^2)\xi = 0. \quad (3.2)$$

Noting that $\xi = \frac{u}{p} = \frac{1}{H}$, we finally conclude that the Euler–Lagrange equation (1.2) is equivalent to the two-dimensional Schrodinger equation

$$\left(\frac{1}{H}\right)_{z\bar{z}} + V \frac{1}{H} = 0 \quad (3.3)$$

with the potential

$$V = 2(\log(uH))_{z\bar{z}} + 2(uH)^2. \quad (3.4)$$

In terms of the variable $\log p = \log(uH) = \phi$ one has

$$V = 2(\phi_{z\bar{z}} + \exp 2\phi). \quad (3.5)$$

Thus for induced Willmore surfaces the linear system (3.3) holds.

Note that for periodic p and p decaying as $|z| \rightarrow \infty$, one has

$$W = 4 \int p^2 dz d\bar{z} = 2 \int V dz d\bar{z}. \quad (3.6)$$

Following the basic idea of the Inverse Scattering Transform method (see [27, 28]), we define deformations via the compatible system of equations

$$\xi_{z\bar{z}} + V\xi = 0,$$

$$\xi_t + A(\partial_z, \partial_{\bar{z}}, V)\xi = 0 \quad (3.7)$$

where A is a differential operator and t is a deformation parameter. The compatibility condition (3.7) is equivalent to nonlinear PDE for V . The simplest nonlinear PDE of this type is the Novikov–Veselov equation ([25])

$$V_t + V_{zzz} + V_{\bar{z}\bar{z}\bar{z}} + 3(V(\partial_{\bar{z}}^{-1}V_z))_z + 3(V(\partial_z^{-1}V_{\bar{z}}))_{\bar{z}} = 0. \quad (3.8)$$

This equation is the first one from the infinite hierarchy of Novikov–Veselov equations which are integrable by the IST method. It is easy to show that for the Novikov–Veselov equation (3.8) as for all equations from this hierarchy the identity

$$\frac{\partial}{\partial t} \int V dz d\bar{z} = 0. \quad (3.9)$$

holds.

Notice that in one-dimensional limit the equation (3.3) reduces to the linear problem for one-dimensional Schrodinger equation and the Novikov–Veselov equation (3.8) reduces to the Korteweg–de Vries equation

$$V_t + \frac{1}{4} V_{xxx} + 12 V V_x = 0. \quad (3.10)$$

Possible geometrical meaning of such deformations will be considered elsewhere.

4. Induced tori of revolution and the mKDV equations

In the rest of the paper we will consider surfaces of revolution that correspond to the case when $p(z, \bar{z}) = p(x)$ where $z = x + iy$ and x and y are real-valued variables, and functions ψ_1 and ψ_2 which define the generalized Weierstrass inducing have the following form

$$\psi_1 = r(x) \exp \frac{iy}{2}, \quad \psi_2 = s(x) \exp \frac{iy}{2} \quad (4.1)$$

where $r(x)$ and $s(x)$ are real-valued functions.

For surfaces of revolution system (2.1) takes the form

$$\begin{aligned} r_x &= -\frac{1}{2}r + 2ps, \\ s_x &= \frac{1}{2}s - 2pr \end{aligned} \quad (4.2)$$

These equation means that the vector function (r, s) belongs to the kernel of the following operator

$$L = \partial_x - \frac{1}{2} \begin{pmatrix} \lambda & v \\ -v & -\lambda \end{pmatrix} \quad (4.3)$$

for

$$\lambda = -1, v = 4p.$$

Since (4.3) we have

$$\begin{aligned}(rs)_x &= -\frac{v}{2}(r^2 - s^2), (r^2 + s^2)_x = \lambda(r^2 - s^2), \\ (r^2 - s^2)_x &= \lambda(r^2 + s^2) + 2vrs.\end{aligned}\tag{4.4}$$

It follows directly from (4.4) that functions u and p are related via the following identity

$$v = \frac{\lambda^2 u - u_{xx}}{\sqrt{\lambda^2 u^2 - u_x^2}}\tag{4.5}$$

which for $\lambda = -1, p = 4v$ has the form

$$p = \frac{u - u_{xx}}{4\sqrt{u^2 - u_x^2}}\tag{4.6}$$

We call the induced surface of revolution one-periodic if the functions $r(x)$ and $s(x)$ are periodic with the same period T . In this case it follows from (2.2) that functions X^1 and X^2 are double-periodic but the formula for X^3 has the form

$$X^3 = -4 \int r(x)s(x)dx\tag{4.7}.$$

Generally formulae (2.2) induce in this case a cylinder of revolution. We also mention that if

$$\int_0^T r(x)s(x)dx = 0\tag{4.8}$$

then in virtue of (4.7) this one-periodic cylinder converts into a torus of revolution. Via (2.3) and (2.4) the total squared mean curvature of such torus M is equal to

$$W(M) = \int_M H^2 d\mu = \int_0^T dx \int_0^{2\pi} dy \frac{p^2}{u^2} 4u^2 = 8\pi \int_0^T p^2 dx\tag{4.9}$$

where $d\mu = 4u^2 dx dy$ is the natural measure on M which is determined the induced metric.

Now let us introduce mKdV hierarchy. We will follow formulae from [29].

Let us consider the following matrix operator:

$$K_{2n+1} = \frac{1}{2} \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\tag{4.10}$$

where matrix elements are determined by following formulae

$$A = \sum_{k=0}^n A_{2k+1}^{(n)} \lambda^{2k+1},$$

$$\begin{aligned}
B + C &= 2 \sum_{k=0}^{n-1} S_{2k+1}^{(n)} \lambda^{2k+1}, B - C = 2 \sum_{k=0}^n T_{2k}^{(n)} \lambda^{2k}, \\
A_{2k+1}^{(n)} &= \partial_x^{-1} (v D^{n-k-1} v_x), A_{2n+1}^{(n)} = 1, \\
S_{2k+1}^{(n)} &= D^{n-k-1} v_x, T_{2k}^{(n)} = \partial_x^{-1} D^{n-k} v_x, \\
D &= \partial_x^2 + v^2 + v_x \partial_x^{-1} v.
\end{aligned} \tag{4.11}$$

The Lax equations

$$[\partial_{t_{2n+1}} + K_{2n+1}, L] = 0 \tag{4.12}$$

form hierarchy of nonlinear equations known as modified Korteweg–de Vries (mKdV) hierarchy. These equations are equivalent to the following

$$v_{t_{2n+1}} = D^n v_x \tag{4.13}$$

and the first nontrivial equation among them is the famous mKdV–equation

$$v_t = \frac{3}{2} v^2 v_x + v_{xxx}. \tag{4.14}$$

One can see that (4.14) is one–dimensional limit of (2.12) (for $p = \frac{v(x)}{4}$) or (3.8).

Deformation (4.13) induces deformation of $r(x, t_1, \dots)$ and $s(x, t_1, \dots)$

$$\eta_{t_{2n+1}} = K_{2n+1} \eta \tag{4.15}$$

where $\eta^T = (r, s)$. Let us show that condition (4.8) is preserved by all these flows.

Let us compute

$$\begin{aligned}
I_n &= \frac{\partial}{\partial t_{2n+1}} \int_0^T r s dx = \int_0^T (r_{t_{2n+1}} s + r s_{t_{2n+1}}) dx = \\
&= \int_0^T \left(\sum_0^{n-1} S_{2k+1}^{(n)} \lambda^{2k+1} (r^2 + s^2) + \sum_0^n T_{2k}^{(n)} \lambda^{2k} (s^2 - r^2) \right) dx.
\end{aligned} \tag{4.16}$$

Integrating (4.16) by parts with using of (4.4) we convert it as follows

$$I_n = \int_0^T \left(\sum_0^{n-1} \lambda^{2k+1} D^{n-k-1} v_x + \sum_0^n \lambda^{2k-1} D^{n-k} v_x \right) (r^2 + s^2) dx.$$

Thus if we will prove that $J_k = \int_0^T (D^k v_x) (r^2 + s^2) dx = 0$ for any $k \geq 0$ then we will prove that $I_n = 0$ for any n .

It is easy to notice that it follows from (4.4) that

$$J_0 = \int_0^T v_x (r^2 + s^2) dx = - \int_0^T v \lambda (r^2 - s^2) dx = 2\lambda \int_0^T (rs)_x dx = 0.$$

Let us compute J_k (we will omit limits of integration):

$$J_k = \int (D^k v_x)(r^2 + s^2) dx =$$

$$\int (\partial_x^2 + v^2 + v_x \partial_x^{-1} v)(D^{k-1} v_x)(r^2 + s^2) dx = F_1 + F_2 + F_3$$

where in virtue of (4.4)

$$F_1 = \int (D^{k-1} v_x)(r^2 + s^2)_{xx} dx =$$

$$\lambda^2 \int (D^{k-1} v_x)(r^2 + s^2) dx + 2\lambda \int vrs(D^{k-1} v - x) dx,$$

$$F_2 = \int v^2(D^{k-1} v_x)(r^2 + s^2) dx,$$

$$F_3 = \int v_x \partial_x^{-1}(v D^{k-1} v_x)(r^2 + s^2) dx = \int \partial_x^{-1}(v D^{k-1} v_x)(r^2 + s^2) dv =$$

$$- \int v^2(D^{k-1} v_x)(r^2 + s^2) dx - \lambda \int v(r^2 - s^2) \partial_x^{-1}(v D^{k-1} v - x) dx =$$

$$- \int v^2(D^{k-1} v_x)(r^2 + s^2) dx - 2\lambda \int vrs(D^{k-1} v - x) dx.$$

Combining all formulae for F_1, F_2 and F_3 we derive that

$$J_k = \lambda^2 J_{k-1}$$

and since $J_0 = 0$ we obtain that $J_k = 0$ for all k . As we mentioned above it immediately follows now that

$$I_n = 0, \quad n \geq 0.$$

Thus we prove that

equations of the mKdV hierarchy (4.13) deform tori, induced via (2.2) and (4.1), into tori.

Let us now show that the total squared mean curvature $W(M)$ is also preserved by these flows. Firstly mention that operator

$$D^+ = \partial_x^2 + v^2 - v \partial_x^{-1} v_x$$

is formally coadjoint to operator D , i.e.

$$\int f \cdot Dg dx = \int D^+ f \cdot g dx.$$

One can see that

$$\partial_x D^+ = D\partial_x.$$

Since $v = 4p$, the identity

$$W(M) = \frac{\pi}{2} \int_0^T v^2 dx \quad (4.17)$$

holds. The derivative of $W(M)$ with respect to t_{2k+1} is equal to

$$\begin{aligned} \frac{2}{\pi} W(M)_{t_{2k+1}} &= 2 \int v(D^k v_x) dx = \int v_x ((D^+)^k v) dx = \\ &= (v(D^+)^k v)|_0^T - \int v \partial_x ((D^+)^k v) dx = \\ &= (v(D^+)^k v)|_0^T - \int v(D^k v) dx. \end{aligned}$$

It is only enough to mention now that the function $v(D^+)^k v$ is periodic and thus we derive that

$$\int v(D^k v) dx = \frac{1}{2} (v(D^+)^k v)|_0^T = 0.$$

Thus we conclude that

the total squared mean curvature W is the first integral of all flows of the mKdV-hierarchy : $W_{t_{2k+1}} \equiv 0$ for $k > 0$.

Now we proceed to Willmore tori. The Euler-Lagrange equation for the Willmore functional in terms of these functions has the following form

$$\frac{1}{u^2} \left\{ \frac{1}{4} \left(\frac{p}{u} \right)_{xx} + \frac{2p^3}{u} + \frac{p}{2u} (\log u)_{xx} \right\} = 0. \quad (4.18)$$

Multiplying (4.18) by $4u^4$ we obtain

$$p_{xx}u + pu_{xx} - 2p_x u_x + 8up^3 = 0. \quad (4.19)$$

Notice that (4.19) follows from (3.1) as its one-dimensional reduction. Differentiating (4.19) by x we obtain

$$u_{xxx}p - u_{xx}p_x + u_x(-p_{xx} + 8p^3) + u(24p^2p_x + p_{xxx}) = 0. \quad (4.20)$$

But also we can differentiate (4.6) by x (for $v = 4p, \lambda = -1$) and conclude that

$$u_{xxx}p - u_{xx}p_x - u_x(1 - 16p^2)p + up_x = 0. \quad (4.21)$$

It follows from (4.20) and (4.21) that for Willmore tori of revolution the important equation

$$(8p^3 + p_{xx} - p)u_x + (p_x - 24p^2p_x - p_{xxx})u = 0. \quad (4.22)$$

holds.

It follows from (4.22) that

$$8p^3 + p_{xx} - p = a \cdot u, a = \text{const}. \quad (4.23)$$

Thus we conclude that

for every induced Willmore surface of revolution equation (4.23) holds.

5. Clifford Torus

Here we demonstrate how the Clifford torus can be obtained by inducing (2.2),(4.1).

Let S^3 be a unit sphere in four-dimensional Euclidean space \mathbf{R}^4 . The Clifford torus in \mathbf{R}^4 is an image of the immersion

$$\mathbf{R}^2 \rightarrow S^4 : (u, v) \rightarrow \left(\frac{\cos u}{\sqrt{2}}, \frac{\sin u}{\sqrt{2}}, \frac{\cos v}{\sqrt{2}}, \frac{\sin v}{\sqrt{2}} \right). \quad (5.1)$$

It is easy to notice that this immersion is double periodic and its image will be an embedded torus.

Let us consider the stereographic projection of S^4 onto the plane $x_4 = -1$ from the pole $(0, 0, 0, 1)$:

$$(x_1, x_2, x_3, x_4) \rightarrow \left(\frac{-2x_1}{x_4 - 1}, \frac{-2x_2}{x_4 - 1}, \frac{-2x_3}{x_4 - 1}, -1 \right). \quad (5.2)$$

We will call the image of the Clifford torus with respect to this projection Clifford (in \mathbf{R}^3 —!) again:

$$(u, v) \rightarrow \left(\frac{2 \cos u}{D}, \frac{2 \sin u}{D}, \frac{2 \cos v}{D} \right) \quad (5.3)$$

where $D = \sqrt{2} - \sin v$. One can compute that the first fundamental form is equal to

$$\frac{4}{D^2}(du^2 + dv^2) \quad (5.4)$$

and the second fundamental form is equal to

$$\frac{2(\sqrt{2} \sin v - 1)}{D^2} du^2 + \frac{2}{D^2} dv^2. \quad (5.5)$$

It follows now that the Gaussian curvature K and the mean curvature H are given by the following formulae

$$K = \frac{\sqrt{2} \sin v - 1}{4}, \quad H = \frac{\sin v}{2\sqrt{2}}. \quad (5.6)$$

Let us now put

$$p(x) = \frac{\sin x}{2\sqrt{2}(\sqrt{2} - \sin x)} \quad (5.7)$$

and

$$u(x) = \frac{1}{\sqrt{2} - \sin x}. \quad (5.8)$$

It is easy to check now by using of direct computations that functions $r(x)$ and $s(x)$ such that

$$r^2 = \frac{u - u_x}{2}, \quad s^2 = \frac{u + u_x}{2}, \quad rs = \frac{\sqrt{2} \sin x - 1}{2(\sqrt{2} - \sin x)^2} \quad (5.9)$$

satisfy system (4.2).

We can also obtain functional equation for $p(x)$. It follows from (5.7) that

$$\sin x = \frac{8p}{1 + 2\sqrt{2}p}. \quad (5.10)$$

Differentiating (5.10) by x we obtain expression for $\cos x$ in terms of p and p_x :

$$\cos x = \frac{8p_x}{(1 + 2\sqrt{2}p)^2}. \quad (5.11)$$

Substituting (5.10) and (5.11) into the trivial identity

$$\sin^2 x + \cos^2 x = 1$$

we obtain

$$p_x^2 = -4p^4 + 2p^2 + \frac{p}{\sqrt{2}} + \frac{1}{16} \quad (5.12)$$

6. mKDV–stationary Willmore tori of revolution

We will restrict ourselves to the Willmore tori generated by stationary solutions of the mKDV equation (4.14) (for $v = 4p$). Such solutions obey the equation

$$c_2 p_x + p_{xxx} + 24p^2 p_x = 0 \quad (6.1)$$

where c_2 is an arbitrary constant. The relation (6.1) implies that

$$p_{xx} + 8p^3 - c_2p - \frac{c_1}{2} = 0 \quad (6.2)$$

or finally

$$p_x^2 = -4p^4 + c_2p^2 + c_1p + c_0 \quad (6.3)$$

where c_0 and c_1 are also arbitrary constants.

It is easy to notice that this function $p(x)$ is elliptic.

Substituting (6.2) into (4.23) we obtain

$$a \cdot u = (c_2 - 1)p + \frac{c_1}{2}. \quad (6.4)$$

Thus we arrive to two possibilities

- 1) $a \neq 0$;
- 2) $a = 0$ and in this case $c_2 = 1, c_1 = 0$.

In the first case substituting (6.4) into (4.6) we obtain that

$$c_2 = 2, c_0 = \frac{4c_1^2 - 1}{16}. \quad (6.5)$$

But one can also substitute (6.4) into (4.19) and obtain that

$$c_1 = c_2d, c_0 = \frac{c_1d}{4} \quad (6.6)$$

where $d = \frac{c_1}{2(c_2-1)}$. Combining (6.5) and (6.6) together we obtain that

$$c_2 = 0, c_1^2 = \frac{1}{2}, c_0 = \frac{1}{16}.$$

But as one can see (5.12) that we obtain Clifford torus (note that functions $p(x)$ and $-p(x)$ induce the same surface). Thus

$a \neq 0$ in (6.4) then the only Willmore surface of revolution for which (6.4) holds is the Clifford torus.

It is left now to consider the case when

$$p_x^2 = -4p^4 + p^2 + \alpha \quad (6.7)$$

where α is a real parameter. This family may contain non-trivial periodic potentials of two different types: 1) for $\frac{-1}{16} < \alpha < 0$ potential $p(x)$ is positive or negative and $\frac{1-\sqrt{1+16\alpha}}{8} \leq p^2 \leq \frac{1+\sqrt{1+16\alpha}}{8}$; 2) potential p varies from $-C$ till C where $C = \sqrt{1 + \frac{1+\sqrt{1+16\alpha}}{8}}$ for $\alpha > 0$. We will show that by different reasons every of these families does not contain Willmore torus of revolution with $W \leq 2\pi^2$.

Let us assume that potential $p(x)$ which satisfies (6.7) induces a one-periodic cylinder of revolution.

It follows from (4.6) that identity (4.7) is equivalent to the following

$$\delta_0 = \int_0^T \frac{u - u_{xx}}{p} dx = 0. \quad (6.8)$$

Integrating (6.8) by parts two times we obtain that

$$\delta_0 = \int_0^T u \left(\frac{1}{p} + \frac{p_{xx}}{p^2} - \frac{2p_x^2}{p^3} \right) dx. \quad (6.9)$$

But in virtue of (6.7),

$$\delta_0 = -2\alpha \int_0^T \frac{u}{p^3} dx. \quad (6.10)$$

Since $u = r^2 + s^2 > 0$ then for $\alpha < 0$ integrand in the right-hand side of (6.10) does not change its sign and we conclude that $\delta_0 \neq 0$ and thus potentials (6.7) with $\alpha < 0$ do not induce tori of revolution.

We will not obtain an analogous statement for potentials (6.7) with $\alpha > 0$ but we will usually show that such potential induces torus then inequality

$$W = 8\pi \int_0^T p^2(x) dx > 2\pi^2. \quad (6.11)$$

holds.

Let us assume that potential $p(x)$ which satisfies (6.7) with $\alpha > 0$ induces a torus via formulae (2.2) and (4.1). Let T be a period of functions $p(x)$, $r(x)$, and $s(x)$. The total squared mean curvature is equal to

$$W = 8\pi \int_0^T p^2 dx = 32\pi \int_0^{\max p} \frac{p^2 dp}{\sqrt{-4p^4 + p^2 + \alpha}}. \quad (6.12)$$

Let us put

$$\beta = \sqrt{1 + 16\alpha}, C = \frac{1 + \beta}{8}, k^2 = \frac{1 + \beta}{2\beta}.$$

It is easy to notice that $\max p = -\min p = \sqrt{C}$ and substituting $v = C - p^2$ into (6.12) we obtain $W = 16\pi I$ where

$$I = \int_0^C \frac{\sqrt{C - v} dv}{\sqrt{v(\beta - 4v)}}. \quad (6.13)$$

But the last formula can be expressed in terms of classic elliptic integrals (see [30], formula 3.141.8):

$$I = \sqrt{\beta} E(k) - \frac{\beta - 1}{2\sqrt{\beta}} F(k) \quad (6.14)$$

where $E(k) = E(\frac{\pi}{2}, k)$, $F(k) = F(\frac{\pi}{2}, k)$, and

$$E(\phi, k) = \int_0^\phi \sqrt{1 - k^2 \sin^2(\tau)} d\tau,$$

$$F(\phi, k) = \int_0^\phi \frac{d\tau}{\sqrt{1 - k^2 \sin^2(\tau)}}.$$

The right-hand side of (6.14) is equal to $f(k) = \frac{E(k) - (1-k^2)F(k)}{\sqrt{2k^2-1}}$ and it is easy to see that this function is well defined for $k \in (1/\sqrt{2}, 1]$ and continuous on this set, $f(1) = 1$ and $f(k) \rightarrow +\infty$ as $k \rightarrow 1/\sqrt{2}$, and the inequality $W > 2\pi^2$ is equivalent to the following

$$f(k) > \frac{\pi}{4}, \frac{1}{\sqrt{2}} < k < 1. \quad (6.15)$$

Let us find the minimum of this function. Since $f(k)$ is smooth, we ought firstly to find points where $f'(k) = 0$. But two following important identities holds ([30], formulae 8.123.2 and 8.124.1):

$$E(k) - (1 - k^2)F(k) = k(1 - k^2) \frac{dF}{dk}, \quad (6.16)$$

$$\frac{d}{dk} \{k(1 - k^2) \frac{dF}{dk}\} = kF. \quad (6.17)$$

In virtue of (6.16) and (6.17) we derive that $f'(k) = 0$ if and only if $2E(k) = F(k)$. The function $F(k)$ is monotonically increasing as $k \rightarrow 1$, and the function $E(k)$ is monotonically decreasing at the same time. We notice now that $2E(k) = F(k)$ for $k^2 \approx 0.826$ ($E \approx 1.1613$, $F \approx 2.3181$ for $k^2 = 0.825$, $E \approx 1.1606$, $F \approx 2.3207$ for $k^2 = 0.826$, and $E \approx 1.1599$, $F \approx 2.3234$ for $k^2 = 0.827$) ([31]). We can make rigorous estimates but from these tables it is rather evident that (5.16) holds because $f(\sqrt{0.826}) \approx 0.9352$ and the estimates given above allow us to make this conclusion (notice also that $\frac{\pi}{4} < 0.7854$).

Thus we conclude that if potential $p(x)$ which satisfies (6.7) with $\alpha > 0$ induces a torus then the total mean curvature of this torus is greater than $2\pi^2$.

Thus we conclude that

if potential $p(x)$ induces, via (2.2) and (4.1), mKdV-invariant Willmore torus of revolution then total squared mean curvature of this torus is greater or equal to $2\pi^2$ and, moreover, is equal to $2\pi^2$ only for the Clifford torus.

The case of mKDV-noninvariant tori will be considered elsewhere.

Acknowledgement: The second author (I.A.T.) was supported by Mathematisches Forschungsinstitut in Oberwolfach and Volkswagenwerk Foundation. This work was done during his stay at the Ruhr-Universitat in Bochum. He also acknowledges partial support by the Russian Foundation for Fundamental Studies (grant 94-01-00528).

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